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Brownian Motion in Granular Gases of Viscoelastic Particles

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Abstract—A theory is developed of Brownian motion in granular gases (systems of many macroscopic particles undergoing inelastic collisions), where the energy loss in inelastic collisions is determined by a restitution coefficient ε. Whereas previous studies used a simplified model with ε = const, the present analysis takes into account the dependence of the restitution coefficient on relative impact velocity. The granular temperature and the Brownian diffusion coefficient are calculated for a granular gas in the homogeneous cooling state and a gas driven by a thermostat force, and their variation with grain mass and size and the restitution coefficient is analyzed. Both equipartition principle and fluctuation–dissipation relations are found to break down. One manifestation of this behavior is a new phenomenon of "relative heating" of Brownian particles at the expense of cooling of the ambient granular gas.

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1. INTRODUCTION

Brownian motion, discovered almost two centuries ago [1], is a fundamental process observed in nature over a wide range of spatial and temporal scales. The stochastic motion of a large (Brownian) particle is driven by random collisions with surrounding much smaller particles. A quantitative theory of this phenomenon was originally developed by Einstein in [2], where the mean square displacement linearly increasing with time was expressed in terms of a macroscopic transport coefficient (diffusion coefficient). The physical ideas put forward by Einstein were further developed by Smoluchowski in [3] and Langevin in [4]. In particular, attempts at constructing a formal mathematical model of Brownian motion stimulated the development of the theory of random processes [3].

Progress in nonequilibrium thermodynamics and statistical mechanics made it possible to formulate an ab initio approach to the theory of Brownian motion, based on microscopic equations of motion for Brownian particles [5]. However, since theories of this kind make use of a number of formal transformations, their application requires additional approximations, which are frequently hard to control [6].

Of special interest is Brownian motion in nonequilibrium and dissipative systems, such as granular media, where a variety of interesting phenomena are observed, including deviations from the Maxwellian velocity distribution [7] and breakdown of the equipartition principle, which are manifested by a temperature difference between Brownian particles and the ambient medium [8, 9]. Moreover, even a qualitative change in the type of motion is possible in such systems: a transition can occur from normal Brownian to nearly ballistic motion of a large particle [10].

Granular media, as systems consisting of many particles (grains) that are in turn macroscopic objects of approximately 10^{-4} to 1 m in size, are widespread in nature [7, 11]. They include dust, sand, gravel, and various powders [11]. Rarefied granular media where the total grain volume is much smaller than the total volume occupied by the system are called granular (dissipative) gases [7, 12–14]. Generally, systems of this kind are astrophysical objects. The best known examples are planetary (e.g., Saturn's) rings, protoplanetary disks, and interstellar dust clouds [15]. Because of gravity, granular gases cannot exist under terrestrial conditions without a driving source. For example, vibrating container walls can be used as such a source to keep the system in a gaseous state under laboratory conditions [16–20]. Under natural conditions, granular gases form in flows with steep gradients: avalanche slides; fast transportation of dry bulk materials; or dust and sand entrainment by moving air, as in the cores of tornadoes or lesser whirlwinds.

In systems of this kind, intergranular collisions result in the loss of kinetic energy to grain internal energy. The energy loss in a collision is characterized by the restitution coefficient

$$\varepsilon = \left|\frac{\mathbf{v}'_{12}\cdot\mathbf{e}}{\mathbf{v}_{12}\cdot\mathbf{e}}\right|,$$

where $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}'_{12} = \mathbf{v}'_1 - \mathbf{v}'_2$ are the pre- and post-collision relative particle velocities, respectively, and $\mathbf{e}$ is the unit vector along the line joining the centers of particles at contact. For simplicity, we treat grains as smooth spheres in this study, so that their rotational degrees of freedom can be ignored. It is obvious that $\varepsilon = 1$ and 0 for perfectly elastic and perfectly inelastic collisions, respectively.

The simplest models of granular media assume that $\varepsilon = \text{const}$. Even though this approximation greatly simplifies modeling and frequently makes it possible to obtain analytical results, models with $\varepsilon \neq \text{const}$ are more realistic. Indeed, both experimental data [21–23] and theoretical analyses [24, 25] of collision processes show that the restitution coefficient must strongly depend on the relative impact velocity. This dependence can be found directly by integrating the equations of motion for colliding particles if the interaction forces between them are known. At low velocities, a weakly deformed grain material behaves as a viscoelastic medium [7, 26, 27] with known viscous and elastic forces [27]. Therefore, the desired dependence can be determined by solving appropriate equations of motion.

In this paper, we develop a theory of Brownian motion of large and heavy particles in an ambient gas of much smaller grains. Collisions between particles obey the laws governing viscoelastic collisions [7, 27]. This model offers a more realistic description of processes that occur in nature and demonstrates a number of new and interesting effects, such as nonmonotonic evolution of the temperature ratio between Brownian particles and the ambient gas and "relative heating" of Brownian particles at the expense of cooling of the ambient granular gas.

To introduce the basic concepts used in this study, a one-component granular gas is considered in Section 2. Section 2.1 provides necessary provides necessary theoretical background on the evolution of a freely cooling granular gas in the homogeneous cooling state. In Section 2.2, we consider a thermostated gas of viscoelastic particles, which has never been examined in previous studies. Section 3 focuses on Brownian dynamics: the Fokker–Planck equation for the velocity distribution of Brownian particles is derived in Section 3.1; the dependence of temperature of Brownian particles on time, particle masses and sizes, and collisional energy loss is described in Section 3.2. Section 3.3 deals with Brownian motion in a thermostated granular gas. Section 4 summarizes the main conclusions of his paper.

2. HOMOGENEOUS GRANULAR GAS

This section introduces some basic concepts required for further study.

2.1. Homogeneous Cooling State

In the absence of external forcing, inelastic collisions between granular gas particles cause a continuous decrease in their kinetic energy. By analogy with the kinetic energy of a conventional gas, the granular temperature $T(t)$ is defined as the second-order moment of the granular velocity distribution $f(\mathbf{v}, t)$:

$$\frac{3}{2}nT(t) = \int d\mathbf{v}\frac{m\mathbf{v}^2}{2}f(\mathbf{v}, t), \tag{1}$$

where m is the grain mass and n is the grain number density. The reference state of a granular gas analogous to the equilibrium state of a conventional gas is the homogeneous cooling state, in which $n = \text{const}$ and $T(t)$ is a monotonically decreasing function [7]. For granular gases of low or moderate density considered here, $f(\mathbf{v}, t)$ obeys the Enskog–Boltzmann equation [7, 28]

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = g_2(\sigma)I(f, f), \tag{2}$$

where σ is the grain diameter and $g_2(\sigma)$ is the radial pair correlation function at contact introduced to take into account the increase in collision frequency due to excluded volume effects [7].

The collision integral in the Enskog–Boltzmann equation is conveniently represented as

$$I(f, f) = \sigma^2 n^2 v_T^{-2}\tilde{I}(\tilde{f}, \tilde{f}),$$

where $v_T = \sqrt{2T(t)/m}$ is the thermal velocity. The dimensionless collision integral $\tilde{I}(\tilde{f}, \tilde{f})$ is a functional of the scaling distribution function

$$\tilde{f}(\mathbf{c}, t) = \frac{v_T^3}{n}f(\mathbf{v}, t)$$

of the reduced velocity $\mathbf{c} = \mathbf{v}/v_T$ [7, 29]:

$$\begin{aligned}\tilde{I}(\tilde{f}, \tilde{f}) = &\int d\mathbf{c}_2 \int d\mathbf{e}\Theta(-\mathbf{c}_{12}\cdot\mathbf{e})|-\mathbf{c}_{12}\cdot\mathbf{e}| \\ &\times(\chi\tilde{f}(\mathbf{c}_1'', t)\tilde{f}(\mathbf{c}_2'', t) - \tilde{f}(\mathbf{c}_1, t)\tilde{f}(\mathbf{c}_2, t)),\end{aligned} \tag{3}$$

where the relative velocity $\mathbf{c}_{12} = \mathbf{c}_1 - \mathbf{c}_2$ is the length of the collision cylinder (whose volume $\sigma^2|\mathbf{c}_{12}|$ determines the corresponding collision probability per unit time under the Stosszahlansatz) [7]; the Heaviside step function $\Theta(x)$ restricts the integral to the pre-collision hemisphere; and the reduced pre-collision velocities $\mathbf{c}_1''$ and $\mathbf{c}_2''$ of the inverse collision (with post-collision velocities $\mathbf{c}_1$ and $\mathbf{c}_2$) are [7, 29]

$$\mathbf{c}_1'' = \mathbf{c}_1 - \frac{(1+\varepsilon)}{2\varepsilon}(\mathbf{c}_{12}\cdot\mathbf{e})\mathbf{e},$$

$$\mathbf{c}_2'' = \mathbf{c}_2 + \frac{(1+\varepsilon)}{2\varepsilon}(\mathbf{c}_{12}\cdot\mathbf{e})\mathbf{e};$$

here, as before, $\mathbf{e}$ is the unit vector connecting the centers of mass of particles at the instant of collision and the factor χ accounts for the length ratio between direct and inverse collision cylinders and the transformation Jacobian between pre- and post-collision velocities [7, 29].

For viscoelastic particles, the restitution coefficient contained in the expressions above depends on their relative impact velocity. According to calculations presented in [7, 29],

$$\begin{aligned}\varepsilon &= 1 - C_1\delta(2u(t))^{1/10}|(\mathbf{c}_{12}\cdot\mathbf{e})|^{1/5} \\ &\quad + C_2\delta^2(2u(t))^{1/5}|(\mathbf{c}_{12}\cdot\mathbf{e})|^{2/5} \pm \ldots,\end{aligned} \tag{4}$$

where $C_1 \approx 1.15$, $C_2 \approx 0.798$, $u(t) = T(t)/T(0)$, and the small parameter δ characterizes the energy loss per collision and depends on the mass, size, and viscoelastic properties of the colliding particles [7, 29].

It is important that the granular velocity distribution deviates from the Maxwellian distribution. The deviation can be quantified by a series expansion of $\tilde{f}(\mathbf{c}, t)$ in Sonine polynomials $S_p(x)$ [30–32]:

$$\begin{aligned}\tilde{f}(\mathbf{c}, t) &= \frac{1}{\pi\sqrt{\pi}}\exp(-c^2) \\ &\times\left(1 + \sum_{p=1}^{\infty} a_p(t)S_p(c^2)(c^2)\right).\end{aligned} \tag{5}$$

The Sonine polynomials $S_p(x)$ are the associated Laguerre polynomials $L_p^m(x)$ with $m = d/2 - 1$, where d is the space dimension ($d = 3$ in the present analysis):

$$S_p(x) = L_p^m(x) = \sum_{n=0}^{p}\frac{(-1)^n(m+p)!}{(m+n)!(p-n)!n!}x^n. \tag{6}$$

They are widely used in the kinetic theory of gases [7, 33] because they are orthogonal with Gaussian weight function:

$$\begin{aligned}&\frac{1}{\pi\sqrt{\pi}}\int d\mathbf{c}\exp(-c^2)S_p(c^2)S'_p(c^2) \\ &= \frac{2\left(p+\frac{1}{2}\right)!}{\sqrt{\pi}p!}\delta_{pp'}.\end{aligned} \tag{7}$$

Thus, the scaling distribution $\tilde{f}(\mathbf{c}, t)$ of reduced granular velocity is completely determined by the coefficients $a_i(t)$. In expansion (5), $a_1(t) = 0$ [7]. The second Sonine approximation a_2 is sufficient for analyzing macroscopic properties, such as granular temperature [34, 35].

Substituting (5) into Eq. (2) and changing to a new time variable τ, we obtain the following system of equations [7, 29]:

$$\begin{aligned}\frac{du}{d\tau} &= -\frac{\sqrt{2}\mu_2}{6\sqrt{\pi}}u^{3/2}, \\ \frac{da_2}{d\tau} &= \frac{\sqrt{2u}}{3\sqrt{\pi}}\mu_2(1+a_2) - \frac{\sqrt{2}}{15\sqrt{\pi}}\mu_4\sqrt{u},\end{aligned} \tag{8}$$

where the moments

$$\mu_p = -\int d\mathbf{c}c^p\tilde{I}(\tilde{f},\tilde{f})$$

of the collision integral can be found analytically [7, 29], say, by means of computer algebra [7]. To first order in δ, we have

$$\mu_2 = \left(1 + \frac{6}{25}a_2\right)(2u)^{1/10}\omega_0\delta + \ldots,$$

$$\mu_4 = 4\sqrt{2\pi}a_2 + \frac{28}{5}\left(1 + \frac{129}{100}a_2\right)(2u)^{1/10}\omega_0\delta + \ldots,$$

$$\omega_0 = 2\sqrt{2\pi}2^{1/10}\Gamma(21/10)C_1 \approx 6.485.$$

System (8) can be solved numerically to find $a_2(t)$ and $u(t)$ at any time. In the limit of $t \longrightarrow \infty$, an analytical solution can be found to first order in δ [7, 29]:

$$u(t) = \left(\frac{t}{\tau_0}\right)^{-5/3} + T_1\delta\left(\frac{t}{\tau_0}\right)^{-11/6},$$

$$a_2(t) = -a_{21}\delta\left(\frac{t}{\tau_0}\right)^{-1/6},$$

where

$$T_1 \approx 3.27, \quad a_{21} = \frac{3}{10}2^{1/5}\Gamma\left(\frac{21}{10}\right) \approx 0.415, \tag{9}$$

and the characteristic cooling time τ_0 is estimated by the relation $\tau_0^{-1} \approx 0.55\delta\tau_c^{-1}(0)$, where

$$\tau_c^{-1}(0) = 4\sqrt{\pi}g_2\sigma^2 n\sqrt{\frac{T(0)}{m}}$$

is the inverse initial mean free time.

2.2. Thermostated Dissipative Gas

Experiments are generally performed on externally (e.g., vibrationally) driven granular gases [16–20], with time-independent granular temperature sustained by energy input to the system via collisions between grains and container walls. In the theory of granular gases, simpler thermostat models more amenable to analysis are used [36]. The present analysis makes use of the simplest one: we assume that the grains are driven (in addition to contact forces) by a random (thermostat) force $\mathbf{F}(t)$ such that

$$\langle F_i(t)F_j(t')\rangle = \delta_{ij}\delta(t-t')m^2\xi_0^2. \tag{10}$$

The Enskog–Boltzmann equation for a dissipative gas stochastically driven by this thermostat force [30],

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = g_2(\sigma)I\left(f, f\right) + \frac{\xi_0^2}{2}\frac{\partial^2}{\partial\mathbf{v}^2}f(\mathbf{v}, t), \tag{11}$$

can be rewritten in terms of the scaling distribution function $\tilde{f}(\mathbf{c}, t)$ as

$$\begin{aligned}&\frac{\partial\tilde{f}}{\partial t} - \frac{1}{v_T}\frac{dv_T}{dt}\left(3\tilde{f} + c\frac{\partial\tilde{f}}{\partial c}\right) \\ &= g_2\sigma^2 n v_T\tilde{I} + \frac{1}{v_T^2}\frac{\xi_0^2}{2}\frac{\partial^2}{\partial\mathbf{c}^2}\tilde{f}.\end{aligned}$$

Multiplying it by c^2 and c^4 and taking the integrals over $\mathbf{c}$ yields the equations

$$\frac{\partial\langle c^2\rangle}{\partial t}+\frac{1}{T}\frac{dT}{dt}\langle c^2\rangle = -\sqrt{\frac{2T}{m}}g_2(\sigma)\sigma^2 n\mu_2+\frac{3m\xi_0^2}{2T},$$

$$\frac{\partial\langle c^4\rangle}{\partial t}+\frac{2}{T}\frac{dT}{dt}\langle c^4\rangle$$
$$= -\sqrt{\frac{2T}{m}}g_2(\sigma)\sigma^2 n\mu_4+10\langle c^2\rangle\frac{m\xi_0^2}{2T}.$$

Setting the time derivatives to zero, we find an expression for the steady-state granular temperature:

$$T = \left(\frac{3}{2\sqrt{2}}\frac{\xi_0^2 m\sqrt{m}}{\mu_2 g_2(\sigma)n}\right)^{2/3}.$$

The corresponding constant value of a_2 is determined by numerical solution of the equation

$$\mu_4 = 5\mu_2, \tag{12}$$

which follows from the two moment equations above. To first order in δ, we have $a_2 = -a_{21}\delta$, where a_{21} has the same value given by (9) as in the asymptotic solution for the homogeneous cooling state.

3. BROWNIAN MOTION

This section presents an analysis of Brownian dynamics.

3.1. Derivation of the Fokker–Planck Equation

We consider the motion of particles of mass $m_b \gg m$ and diameter σ_b in a dissipative gas with particle mass m and diameter σ in the homogeneous cooling state. We define the small parameter $\Delta = m/m_b$ and assume that the number density $n_b \ll n$ of the Brownian particles is sufficiently low that the interaction between them is negligible. Denoting the velocity of a Brownian particle by $\mathbf{v}_b$, we introduce the relative impact velocity $\mathbf{g} = \mathbf{v}_b - \mathbf{v}$ between Brownian and granular particles. The restitution coefficient $\varepsilon_b(\mathbf{g})$ for collisions between them can be derived from the general solution for colliding viscoelastic particles [7] and represented in a form similar to (4):

$$\varepsilon_b(\mathbf{g}) = 1-C_1\delta_b(2u)^{1/10}|(\mathbf{g}^*\cdot\mathbf{e})|^{1/5} + C_2\delta_b^2(2u)^{1/5}|(\mathbf{g}^*\cdot\mathbf{e})|^{2/5}\pm\ldots, \tag{13}$$

where the parameter δ_b characterizes the energy loss per collision and depends on the material properties, masses, and sizes of the Brownian and granular particles [7]; $\mathbf{g}^* = \mathbf{g}/v_T$ is the reduced relative velocity between them.

The evolution of the Brownian velocity distribution $f_b(\mathbf{v}_b)$ is described by the Enskog–Boltzmann equation [7]

$$\frac{\partial f_b(\mathbf{v}_b, t)}{\partial t} = I(f_b, f) \tag{14}$$

with collision integral between Brownian and granular particles represented as

$$I(f_b, f) = \sigma_0^2 g_{2b}(\sigma_0)\int d\mathbf{v}\int d\mathbf{e}\Theta(-\mathbf{g}\cdot\mathbf{e})|\mathbf{g}\cdot\mathbf{e}| \times(\chi_b f_b(\mathbf{v}_b'', t)f(\mathbf{v}'', t)-f_b(\mathbf{v}_b, t)f(\mathbf{v}, t)).$$

Here, $g_{2b}(\sigma_0)\equiv g_2(\sigma_0)$ is the radial pair correlation function at contact between Brownian and granular particles, $\sigma_0 = (\sigma_b+\sigma)/2$, χ_b is similar in meaning to the analogous factor in (3), and pre-collision velocities $\mathbf{v}_b''$ and $\mathbf{v}''$ are related to post-collision velocities $\mathbf{v}_b$ and $\mathbf{v}$ as

$$\mathbf{v}_b'' = \mathbf{v}_b-\frac{\delta\mathbf{v}_b}{\varepsilon_b},$$

$$\mathbf{v}'' = \mathbf{v}+\frac{\delta\mathbf{v}_b}{\varepsilon_b\Delta},$$

with

$$\delta\mathbf{v}_b = \frac{\Delta}{1+\Delta}(1+\varepsilon_b)(\mathbf{g}\cdot\mathbf{e})\mathbf{e}. \tag{15}$$

Following an approach commonly used to derive the Fokker–Planck equation [7, 8], we take an arbitrary function $H(\mathbf{v}_b)$ and consider the integral

$$I[H] = \int d\mathbf{v}_b H(\mathbf{v}_b)I(f_b, f). \tag{16}$$

Performing a change of variables, we obtain

$$I[H] = \sigma_0^2 g_2(\sigma_0)\int d\mathbf{v}_b\int d\mathbf{v}\int d\mathbf{e}\Theta(-\mathbf{g}\cdot\mathbf{e})|\mathbf{g}\cdot\mathbf{e}| \times f_b(\mathbf{v}_b, t)f(\mathbf{v}, t)(H(\mathbf{v}_b-\delta\mathbf{v}_b)-H(\mathbf{v}_b)). \tag{17}$$

Noting that the change in the velocity of a Brownian particle caused by a collision is small because of its relatively large mass, we substitute the Taylor series expansion of $H(\mathbf{v}_b-\delta\mathbf{v}_b)$ in $\delta\mathbf{v}_b$, as given by (15), perform the integral, and compare the resulting expression with (16) to rewrite the Enskog–Boltzmann collision integral as

$$I(f_b, f) = \frac{\partial}{\partial\mathbf{v}_b}\left(\gamma(t)\mathbf{v}_b+\tilde{\gamma}(t)\frac{\partial}{\partial\mathbf{v}_b}\right)f_b, \tag{18}$$

where

$$\gamma(t)\mathbf{v}_b = \sigma_0^2 g_2(\sigma_0)\int d\mathbf{v}_b\int d\mathbf{e}\mathbf{e}\Theta(-\mathbf{g}\cdot\mathbf{e})|\mathbf{g}\cdot\mathbf{e}| \times(\mathbf{g}\cdot\mathbf{e})f(\mathbf{v}, t)\frac{\Delta}{1+\Delta}(1+\varepsilon_b(\mathbf{g})), \tag{19}$$

$$\tilde{\gamma}(t)\delta_{ij} = \frac{1}{2}\sigma_0^2 g_2(\sigma_0)\int d\mathbf{v}_b\int d\mathbf{e}e_ie_j\Theta(-\mathbf{g}\cdot\mathbf{e}) \times|\mathbf{g}\cdot\mathbf{e}|(\mathbf{g}\cdot\mathbf{e})^2 f(\mathbf{v}, t)\left(\frac{\Delta}{1+\Delta}(1+\varepsilon_b(\mathbf{g}))\right)^2. \tag{20}$$

It is obvious that Eq. (14) with collision integral (18) is the Fokker–Planck equation

$$\frac{\partial f_b}{\partial t} = \frac{\partial}{\partial \mathbf{v}_b}\left(\gamma(t)\mathbf{v}_b + \tilde{\gamma}(t)\frac{\partial}{\partial \mathbf{v}_b}\right)f_b. \qquad (21)$$

Using expression (13) for the restitution coefficient, we expand (19) and (20) in powers of $((T_b/T)\Delta)^{1/2}$ (assumed to be a small parameter) and perform integration to obtain

$$\gamma(t) = \gamma_0 u^{1/2}\left[\left(1-\frac{1}{8}a_2\right) - \gamma_1\delta_b u^{1/10}\left(1-\frac{3}{25}a_2\right) + \gamma_2\delta_b^2 u^{1/5}\left(1-\frac{21}{200}a_2\right) - \dots\right], \qquad (22)$$

$$\tilde{\gamma}(t) = \tilde{\gamma}_0 u^{3/2}\left[\left(1+\frac{3}{8}a_2\right) - \tilde{\gamma}_1\delta_b u^{1/10}\left(1+\frac{12}{25}a_2\right) + \tilde{\gamma}_2\delta_b^2 u^{1/5}\left(1+\frac{119}{200}a_2\right) - \dots\right]. \qquad (23)$$

These expressions contain the coefficients γ_0 and $\tilde{\gamma}_0$ in the Fokker–Planck equations for elastic Brownian particles,

$$\gamma_0 = \frac{2}{3}\sqrt{2}\Delta\frac{\sigma_0^2 g_2(\sigma_0)}{\sigma^2 g_2(\sigma)}\tau_c^{-1}(0),$$

$$\tilde{\gamma}_0 = \frac{T(0)}{m_b}\gamma_0,$$

and the constant parameters

$$\gamma_1 = \frac{11}{200}\Gamma\left(\frac{1}{10}\right)C_1 2^{1/10} \approx 0.647,$$

$$\gamma_2 = \frac{3}{25}\Gamma\left(\frac{1}{5}\right)C_2 2^{1/5} \approx 0.505,$$

$$\tilde{\gamma}_1 = 2\gamma_1 \approx 1.294,$$

$$\tilde{\gamma}_2 = \frac{1}{2}\frac{C_1^2 + 4C_2}{C_2}\gamma_2 \approx 1.431.$$

When $t \gg \tau_c(0)$, the coefficients γ and $\tilde{\gamma}$ are the following functions of time:

$$\gamma(t) = \gamma_0\left(\frac{t}{\tau_0}\right)^{-5/6}\left(1 + \left(\frac{t}{\tau_0}\right)^{-1/6} \times \left(\frac{1}{2}T_1\delta + \frac{1}{8}a_{21}\delta - \gamma_1\delta_b\right) + \dots\right), \qquad (24)$$

$$\tilde{\gamma}(t) = \tilde{\gamma}_0\left(\frac{t}{\tau_0}\right)^{-5/2}\left(1 + \left(\frac{t}{\tau_0}\right)^{-1/6} \times \left(\frac{3}{2}T_1\delta - \frac{3}{8}a_{21}\delta - \tilde{\gamma}_1\delta_b\right) + \dots\right). \qquad (25)$$

These results demonstrate that the fluctuation–dissipation relation between $\gamma(t)$ and $\tilde{\gamma}(t)$ breaks down for a granular gas in the homogeneous cooling state. Indeed, whereas

$$\tilde{\gamma} = (T/m_b)\gamma$$

for a system in equilibrium [7, 35], this relation obviously does not hold unless $\delta = 0$.

3.2. Evolution of the Temperature of Brownian Particles

The integral over $\mathbf{v}_b$ of Eq. (21) multiplied by $m v_b^2/2$ yields an evolution equation for the kinetic temperature analogous to the granular temperature defined by (1):

$$\frac{dT_b}{dt} = -2\gamma(t)T_b(t) + 2\tilde{\gamma}(t)m_b. \qquad (26)$$

The solution to this linear equation is

$$T_b(t) = T_b(t_0)\exp\left(-\int_{t_0}^{t}2\gamma(\tau)d\tau\right) + 2m_b\int_{t_0}^{t}dt_1\tilde{\gamma}(t_1)\exp\left(-\int_{t_1}^{t}2\gamma(\tau)d\tau\right). \qquad (27)$$

The asymptotic time dependence of the temperature of Brownian particles at $t \gg \tau_0$ can be found for small δ by substituting (24) and (25) into (27) and noting that an exponential decays much faster than any power. Simple calculations show that

$$\frac{T_b(t)}{T(0)} = \left(\frac{t}{\tau_0}\right)^{-5/3} + T_{b1}\delta\left(\frac{t}{\tau_0}\right)^{-11/6} + \dots, \qquad (28)$$

where

$$T_{b1} = T_1 - \frac{1}{2}a_{21} + (\gamma_1 - \tilde{\gamma}_1)\frac{\delta_b}{\delta}.$$

Assuming for simplicity that Brownian and granular particles are similar in terms of dissipative properties ($\delta_b = \delta$), we have $T_{b1} \approx 2.413$.

The coincidence of the temperature of Brownian particles and the temperature of the surrounding granular gas in the zero-order approximation with respect to δ is fully consistent with the well-known fact that the limit values approached by the characteristics of a granular gas of viscoelastic particles at infinite time correspond to those of a gas of elastic particles [7, 29, 34]. However, it is clear from (28) that the values of T

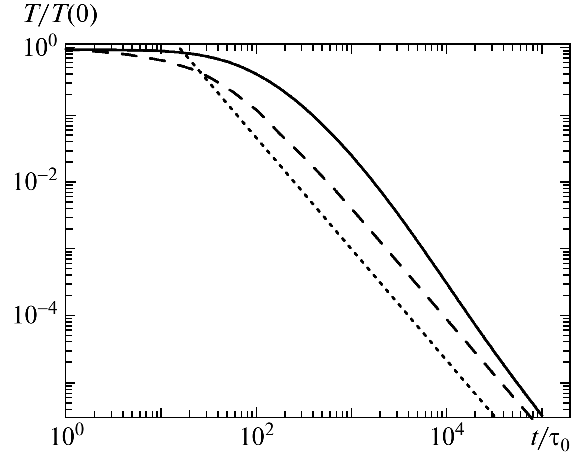

Fig. 1. Evolution of temperature $T_b(t)$ of Brownian particles (solid curve) and granular temperature $T(t)$ (dashed curve) in an undriven granular gas. Dotted curve represents the asymptotic law $t^{-5/3}$. The Brownian-to-granular mass ratio is $m_b/m = 5000$; collisional energy loss is characterized by $\delta = 0.05$. For simplicity, both ambient-gas and Brownian-particle number densities are sufficiently low that $g_2(\sigma) \approx g_2(\sigma_0) \approx 1$.

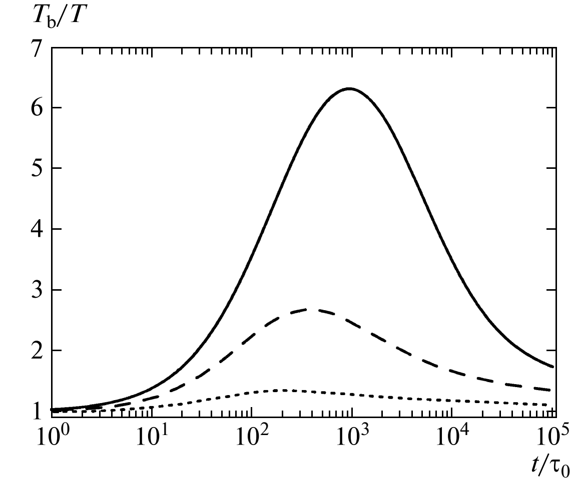

Fig. 2. Evolution of the ratio between temperature $T_b(t)$ of Brownian particles and granular temperature $T(t)$ for δ = 0.05 (solid curve), 0.03 (dashed curve), and 0.01 (dotted curve); $m_b/m = 5000$. For simplicity, both ambient-gas and Brownian-particle number densities are sufficiently low that $g_2(\sigma) \approx g_2(\sigma_0) \approx 1$.

and T_b calculated up to the first order with respect to δ are different; i.e., energy equipartition is violated, as in the case of a constant restitution coefficient [8, 10].

The temperature of Brownian particles at any time can be found by numerical solution of Eq. (26). The solid and dashed curves in Fig. 1 represent the temperatures of Brownian particles and the ambient granular temperature, respectively. The equal log–log slopes of these curves at long times correspond to the asymptotic power law $T_b(t) \propto t^{-5/3}$ predicted by (28). Note that slower cooling of the impurity, as compared to the ambient granular gas, can be interpreted as "relative heating" of Brownian particles.

This phenomenon is further illustrated by the evolution of the impurity-to-gas temperature ratio shown in Figs. 2 and 3. Initially, the granular gas and Brownian particles have similar velocity distributions and temperatures. In the course of time, the temperature of Brownian particles increases relative to the granular temperature, the ratio reaches a maximum, and then the temperatures converge as collisions become increasingly elastic. The temperature difference between particles of different kinds increases with increasing dissipation, since the granular gas cools faster than the Brownian particles because of their larger mass.

Figure 3 demonstrates that the deviation from energy equipartition increases with the Brownian-to-granular mass ratio, while the temperature convergence begins earlier as the ratio decreases. Note that the dependence of the temperature ratio variation on mass is weaker than its dependence on dissipation parameters. Indeed, whereas the thermal inertia of Brownian particles increases with their mass, the corresponding increase in their size at constant density implies faster energy transfer between gas and impurity particles due to higher frequency of contacts between them.

Furthermore, a detailed analysis shows that "absolute heating" of Brownian particles occurs in a certain range of parameter values. This heat transfer from a cooler subsystem to a warmer one by no means violates the second law of thermodynamics, because granular gases are thermodynamically open systems.

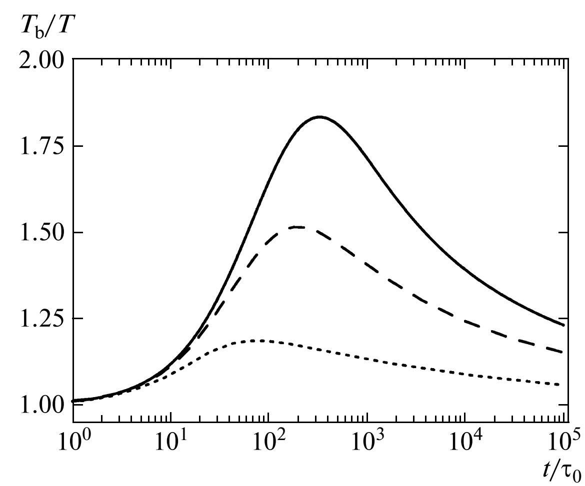

Fig. 3. Evolution of temperature ratio $T_b(t)/T(t)$ for m_b/m = 500 (dotted curve), 4000 (dashed curve), and 10000 (solid curve); $\delta = 0.015$. As above, it is assumed that $g_2(\sigma) \approx g_2(\sigma_0) \approx 1$.

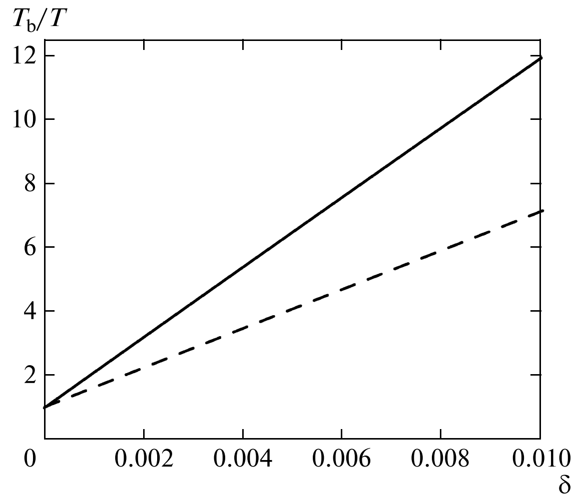

Fig. 4. Brownian-to-granular temperature ratio (Eq. (31)) vs. dissipative parameter δ for $m_b/m = 100$ (dashed curve) and 150 (solid curve). As above, it is assumed that $g_2(\sigma) \approx g_2(\sigma_0) \approx 1$.

3.3. Brownian Motion in a Steady-State Granular Gas

Now, we consider Brownian motion in a steady-state dissipative gas driven by random force (10), while the impurity particles are driven by a force $\mathbf{F}_b(t)$ with the following correlation function:

$$\langle F_{bi}(t)F_{bj}(t')\rangle = \delta_{ij}\delta(t-t')m_b^2\xi_b^2.$$

In this case, the equation of evolution of the velocity distribution function for Brownian particles contains an additional term representing the driving effect of $\mathbf{F}_b(t)$:

$$\frac{\partial f_b}{\partial t} = \frac{\partial}{\partial \mathbf{v}_b}\left(\gamma \mathbf{v}_b + \Gamma\frac{\partial}{\partial \mathbf{v}_b}\right)f_b, \tag{29}$$

where

$$\Gamma = \left(\tilde{\gamma} + \frac{\xi_b^2}{2}\right).$$

The coefficients γ and $\tilde{\gamma}$ are given by expansions similar to (22) and (23) for a cooling gas, but are constant in the steady-state case since $u = 1$, and the coefficient a_2 is determined by solving Eq. (12). Setting all time derivatives to zero and following the derivation of (26) for the homogeneous cooling state, we obtain the following expression for the steady-state temperature of Brownian particles:

$$T_b = m_b\left(\frac{\Gamma}{\gamma}\right) = m_b\frac{\tilde{\gamma} + \xi_b^2/2}{\gamma}. \tag{30}$$

This expression shows that a fluctuation–dissipation relation holds in the case of a thermostated granular gas. Note that its validity for a system in a steady state, but not in thermodynamic equilibrium, is far from obvious. Moreover, this state is characterized by a temperature difference between Brownian and granular particles (see below).

In the limit of weak dissipation, an asymptotic expression for the temperature of Brownian particles is obtained by expansion of (30) in powers of δ and δ_b:

$$T_b = T\left(1 - (\tilde{\gamma}_1 - \gamma_1)\delta_b - \frac{1}{2}a_{21}\delta\right) + \frac{3}{16}\frac{1}{\sqrt{2\pi}}$$
$$\times \frac{m_b^2}{m}\frac{1}{\sigma_0^2 g_2(\sigma_0)}\sqrt{\frac{m}{T}}\frac{\xi_b^2}{n}\left(1 - \frac{1}{8}a_{21}\delta + \gamma_1\delta_b\right).$$

When the random force per unit mass is equal for the Brownian and granular particles ($\xi_b = \xi_0$) and so are the dissipation parameters ($\delta_b = \delta$), the temperature of Brownian particles is

$$\frac{T_b}{T} = 1 + \left(-\gamma_1 - \frac{1}{2}a_{21} + \frac{2^{1/10}}{8\sqrt{\pi}}\left(\frac{m_b}{m}\right)^2\frac{g_2(\sigma)\sigma^2}{g_2(\sigma_0)\sigma_0^2}\omega_0\right)\delta + \ldots. \tag{31}$$

The deviation from energy equipartition is much stronger for a steady-state granular gas driven by a random force imparts the same acceleration to all particles than for a gas in the homogeneous cooling state. The temperature T_b is a rapidly increasing function of Brownian particle mass and dissipation parameters, as illustrated by Fig. 4.

The diffusion coefficient for Brownian particles can easily be calculated by noting that Eq. (29) is a standard Fokker–Planck equation with constant coefficients and the corresponding Langevin equation is written as follows [7, 37]:

$$\frac{d\mathbf{v}_b}{dt} = -\gamma\mathbf{v}_b - \mathcal{F}(t),$$

$$\langle\mathcal{F}(t)\rangle = 0, \quad \langle\mathcal{F}(t)\mathcal{F}(t')\rangle = \Gamma\delta(t'-t).$$

From these equations, we determine the velocity autocorrelation function for Brownian particles and use the Green–Kubo relations to obtain [7]

$$D_b = \frac{T_b}{m_b\gamma}. \tag{32}$$

This result is similar to an expression for molecular gas diffusivity. However, an important difference lies in the fact that the temperature of Brownian particles is not equal to the steady-state granular temperature obtained when the energy loss in inelastic collisions is balanced by the energy input from the heat bath.

4. CONCLUSIONS

The evolution of a granular gas containing large impurity (Brownian) particles is analyzed. As distinct from previous studies, where a simplified model with constant restitution coefficient was employed, our analysis concerns a realistic coefficient of restitution calculated for colliding viscoelastic particles.

Our results reveal breakdown of the equipartition principle, which manifests itself by a significant temperature difference between Brownian particles and the surrounding gas. This effect is predicted for both freely cooling and thermostated gases. At the same time, the fluctuation–dissipation relation between the effective friction coefficient and the random force amplitude, always violated in the homogeneous cooling state, is valid for a thermostated gas despite the temperature difference between the Brownian particles and the gas.

A remarkable new phenomenon predicted in the present study is the "relative heating" of Brownian particles at the expense of cooling of the ambient gas. However, it is a transient phenomenon: the temperatures of Brownian and granular particles converge in the long-time asymptotic limit. In a sense, it resembles the "cooling" of particles in billiards [38], where a decrease in the velocity of fast particles is caused by the motion of infinitely heavy billiard walls.

It should also be noted that this phenomenon does not imply any violation of the second law of thermodynamics, because granular gases are thermodynamically open systems. Indeed, inelastic collisions provide a mechanism for continual transfer of (mechanical) energy from the degrees of freedom treated explicitly (particle coordinates and velocities) to the microscopic degrees of freedom associated with the atomic structure of particles, whose characteristics are hidden in the dissipative parameters of the grain material.

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